Some excercises for the ESI autumn school

Differential Equations, Geometry and Arithmetic

October 7-11, 2024

1) Make sketches for the real part of the curves $E_t : y^2 = x(x-1)(x-t)$ for $t \in \mathbb{R}$. For what values of t is the curve E_t singular? What happens for $t = \infty$?

2) Show that the period of the elliptic curve E_t

$$\phi(t) = \int_0^t \frac{dx}{\sqrt{x(x-1)(x-t)}} = \int_0^t \frac{dx}{\sqrt{x(1-x)(1-xt)}}$$

and by expansion of the right hand side show that

$$\phi(t) = \pi(1 + (\frac{1}{2})^2 t + (\frac{1 \cdot 3}{2 \cdot 4})^2 t^2 + (\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6})^2 t^3 + \ldots$$

Note that

$$\frac{1}{\sqrt{1-t}} = 1 + (\frac{1}{2})t + (\frac{1 \cdot 3}{2 \cdot 4})t^2 + (\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6})t^3 + \ldots$$

and that

$$\binom{-1/2}{k} = \frac{1}{4^k}\binom{2k}{k}.$$

3) What is the differential equation satisfied by $I(t)$? What is the Riemann symbol of that operator? What is the local behaviour of the solutions around each of the singular points? What are the local monodromies? Can you find the global monodromy representation?

4) Show that the functions $\phi_{\alpha,k} := t^\alpha \log^k t, k = 0, ..., n-1$, holomorphic on a slit disc, form a base of solutions to the DE $(\theta - \alpha)^n \phi = 0$. What is the monodromy matrix around 0 for this basis?

5) Check that $t^n \frac{d^n}{dt^n} = \theta(\theta-1)\cdots(\theta-n+1)$.

6) What are the exponents of the general hypergeometric operator at $t=1$? So what is the complete Riemann symbol?

7) Verify the Deuring formula for some values of p and t, say, $p=11, t=3$.

8) Give a proof of Deurings formula. You may procede along the following lines: for p prime, $p \neq 2$, $f \in \mathbb{F}_p[x]$ we set $C := \{(x,y) \in \mathbb{F}_p \times \mathbb{F}_p \mid y^2 = f(x)\}$.

(i) Show that

$$\#C = \sum_{x \in \mathbb{F}_p} (1 + f(x)^{\frac{p-1}{2}}) = \sum_{x \in \mathbb{F}_p} f(x)^{\frac{p-1}{2}} \mod p$$

(ii) If the degree of $f \leq 3$, then

$$\#C = -[f(x)^{\frac{p-1}{2}}]_{p-1} \mod p$$

so that for $f(x) = x(x-1)(x-t)$ we obtain

$$\#C = -(-1)^{\frac{p-1}{2}} \sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k}^2 t^k$$

(iii) Show that $\binom{\frac{p-1}{2}}{k} = \binom{\frac{-1}{2}}{k} \mod p$, thus completing the proof of Deurings formula.

(iv) Show that in fact $\binom{\frac{-1}{2}}{k} = 0 \mod p$ for $\frac{p-1}{2} < k \leq p-1$.